Reverting Black-Litterman: What Views are Consistent with Portfolio Weights?

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Abstract

If a variety of quantitative and qualitative methods is used to construct a portfolio, it becomes difficult to identify the final views about future market movements that have led to a particular portfolio. This paper shows that by a simple rearrangement of the Black-Litterman model implicit views can be extracted from any portfolio. The analysis of these implicit views is important for the reevaluation of portfolio allocations and for communicational purposes. Moreover, this paper provides an example where implicit views are used to build a portfolio with a different risk while leaving Black-Litterman views unchanged.

Keywords: Black-Litterman model, investment management, portfolio theory

JEL-Classification: G11, G12, G14

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1 Introduction

Investors and asset managers who calculate portfolio allocations using traditional mean-variance optimization generally face the problem of unstable and economically unintuitive weights in resulting portfolios. The instability stems from the input parameters used in mean-variance optimization – i.e. the assets' expected returns, variances, and covariances. While Markowitz's (1952) traditional portfolio theory assumes known expected returns and a known covariance matrix, in practice they must be estimated. Estimation error – mainly in expected returns[1] – has however strong impact on resulting portfolios often leading to extreme allocations.

To avoid the difficulty arising from estimation error, Black and Litterman (1992) suggest a method leading to stable and economically meaningful portfolio weights. Their model provides expected returns that contain information about equilibrium returns and incorporate analysts' views about future market movements. Plugging this posterior mean of expected returns into the mean-variance optimizer results in stable and intuitive portfolio allocations. The Black-Litterman (B-L) model has at least two advantages from an asset manager's perspective: First, it leads to stable and intuitive portfolios which do not deviate too much from the reference portfolio. Second, the model allows for incorporating subjective analyst views into a mean-variance optimization framework and thus, combines the knowledge and abilities of both, quantitative and qualitative asset managers.

While the B-L model is used to obtain portfolio weights that are consis-

[1] The importance of estimation error in expected returns is discussed in e.g. Merton (1980) and Michaud (1989). Chopra and Ziemba (1993) and Kan and Zhou (2007) show numerically that estimation error in expected returns has larger impact on portfolio weights than estimation error in the covariance matrix.

tent with equilibrium returns and analyst views, this paper shows that by a simple rearrangement, the application of the B-L model can be reverted. When the composition of the market portfolio and uncertainty in views are known, for any portfolio the implicit B-L views can be expressed. Moreover, it is shown how to express the implicit views of different analysts influencing the allocation of one particular portfolio when all having mutually exclusive areas of expertise (e.g. in different sectors or regions).

Clearly, when asset allocation is solely based on the B-L model, views are known. However, asset allocation often includes different quantitative methods combined with qualitative inputs. In such cases the resulting final views are less transparent or even unknown. The following two examples demonstrate the importance of knowing the views that are implicit in a portfolio.

Within a product family, one might wish to have a single opinion about market movements. As, however, portfolio allocations differ between risk profiles of the same product family, market views can unintendedly change (and not only the risk aversion). On the one hand, expressing the implicit views behind a portfolio helps to analyze the asset allocation decision - i.e. whether a portfolio allocation has changed due to a reassessment of future market movements or due to the change of the portfolio's risk aversion. On the other hand, relevant for portfolio engineering, implicit views extracted from a portfolio can be used to construct a portfolio with a different risk profile without altering Black-Litterman views.

Another example shows the relevance of subdividing implicit views within a portfolio. Consider a portfolio where assets are diversified across two or more sectors. While the composition of the portfolio as a whole may be intuitive, implicit views within one particular sector can be unreasonable.

Expressing these views can thus detect some unwanted portfolio allocations and facilitate the "fine tuning" of a portfolio.

The rest of the paper is structured as follows: Section 2 provides a short review of the Black-Litterman model and derives the key formulas for the reverted B-L model. Section 3 provides two examples about the application of the model. In Section 4, we examine the sensitivity of results with respect to changes in input variables. Section 5 concludes.

2 The Reverted Black-Litterman Model

The B-L model combines two sources of information: equilibrium excess returns from market data and subjective analyst views. Consider a universe of n assets with returns $\boldsymbol{r}$ (bold letters denote vectors or matrices while normal letters are used for scalars). Equilibrium excess returns (i.e. expected market returns minus the riskfree rate) are found by reverse optimization[2]

$$\boldsymbol{\pi} = \gamma_m \boldsymbol{\Sigma} \boldsymbol{w}_m \,,$$

where $\boldsymbol{\Sigma}$ is a positive definite $(n \times n)$ covariance matrix of assets' returns and $\boldsymbol{w}_m$ is an n-vector describing the market portfolio.[3] The coefficient γ_m is the risk aversion coefficient of an investor (with no B-L views) having quadratic utility and investing solely into the market portfolio. In practical terms, γ_m is the risk premium of the market portfolio with risk being defined as portfolio variance.

The second component of the B-L model are analysts' prior views[4] de-

[2]For a derivation of the Black-Litterman model see Satchell and Scowcroft (2000). For more details on the implemenation of the B-L model see Lee (2000), Drobetz (2001), Zimmermann, Drobetz, and Oertmann (2002).

[3]Alternatively, $\boldsymbol{w}_m$ is sometimes considered as a reference or benchmark portfolio.

[4]While $\mathrm{E}(\boldsymbol{r})$ is used in the literature we use $\boldsymbol{r_e}$ to point out the stochastic nature of expected excess returns.

scribed by three parameters $\boldsymbol{P}$, $\boldsymbol{q}$ and $\boldsymbol{\Omega}$

$$\boldsymbol{P r_e} \sim \mathcal{N}(\boldsymbol{q}, \boldsymbol{\Omega}), \tag{1}$$

where $\boldsymbol{P}$ has the dimension $(m \times n)$, $\boldsymbol{r_e}$ is an n-vector of expected excess returns, $\boldsymbol{q}$ is an m-vector and $\boldsymbol{\Omega}$ is an $(m \times m)$ diagonal matrix. According to (1), an analyst has the prior view that the linear combination of expected asset excess returns $\boldsymbol{P r_e}$ is centered around $\boldsymbol{q}$. For example $P = [1 \;\; 0 \;\; \dots \;\; 0]$ (i.e. $m = 1$) and $q = 0.01$ means that the prior view is that the expected excess return of asset 1 is 1 per cent.

The diagonal matrix $\boldsymbol{\Omega}$ contains the uncertainty in prior views. The analyst may believe with 90% probability that the performance of asset 1 will be between 0.5% and 1.5%. With the assumption of normality given in (1), $\boldsymbol{\Omega}$ is determined from the confidence interval $1.645 \times \Omega_{1,1}^{1/2} = 1\% - 0.5\%$, thus, here $\boldsymbol{\Omega} \equiv \Omega_{1,1} = (0.003)^2$.

To fully specify the Bayesian framework used in B-L it is assumed that equilibrium excess returns $\boldsymbol{\pi}$ are centered around analysts' expectations with $\mathcal{N}(\boldsymbol{r_e}, \tau\boldsymbol{\Sigma})$, where τ is a known scaling factor and $\boldsymbol{\Sigma}$ is the returns' covariance matrix. Black and Litterman (1992) argue that τ should be small, as uncertainty in means is smaller than uncertainty in the returns. Drobetz (2002) shows that in practical applications $\tau = 0.3$ provides robust results.

The probability density function (pdf) of posterior expected returns can be written as

$$\text{pdf}(\boldsymbol{r_e}|\boldsymbol{\pi}) \propto \exp\left\{-\frac{1}{2}(\boldsymbol{\pi} - \boldsymbol{r_e})'(\tau\boldsymbol{\Sigma})^{-1}(\boldsymbol{\pi} - \boldsymbol{r_e}) - \frac{1}{2}(\boldsymbol{P r_e} - \boldsymbol{q})'\boldsymbol{\Omega}^{-1}(\boldsymbol{P r_e} - \boldsymbol{q})\right\}. \tag{2}$$

It combines information from equilibrium returns (first summand within the exp operator) and the prior (second summand). The main result of the

Black-Litterman model is the expression for the mean of (2) given by

$$\mathrm{E}(\boldsymbol{r_e^*}) = \left[(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P'}\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right]^{-1}\left[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{P'}\boldsymbol{\Omega}^{-1}\boldsymbol{q}\right], \tag{3}$$

where $\boldsymbol{r_e^*}$ are posterior expectations about assets' expected returns. From (3) is clear that the posterior expectation $\mathrm{E}(\boldsymbol{r_e^*})$ incorporate information from equilibrium excess returns $\boldsymbol{\pi}$ and subjective analyst views given by $\boldsymbol{P}$, $\boldsymbol{q}$ and uncertainty $\boldsymbol{\Omega}$.

Knowing the mean of posterior expectations given in (3) and when no short selling constraints are imposed, a portfolio $\boldsymbol{w}_i$ is constructed by mean variance optimization

$$\boldsymbol{w}_i = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\mathrm{E}(\boldsymbol{r_e^*}). \tag{4}$$

Here, γ denotes the investor specific risk aversion (which is in general not equal to γ_m defined before). With (3) and some transformations shown in Appendix A, the resulting B-L portfolio can be written as

$$\boldsymbol{w} = \frac{\gamma_m}{\gamma}\boldsymbol{w}_m + \frac{1}{\gamma}\left[\boldsymbol{P'}\left(\frac{1}{\tau}\boldsymbol{\Omega} + \boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P'}\right)^{-1}(\boldsymbol{q} - \boldsymbol{P}\boldsymbol{\pi})\right]. \tag{5}$$

Equation (5) shows that those assets' weights for which no opinion was given – i.e. the corresponding elements in $\boldsymbol{P}$ are zero – have the same weights as in the market portfolio if the investor's relative risk aversion γ equals to the "market risk aversion" γ_m. This feature of the B-L model is convenient for practical purposes; only the weights for which opinions are given deviate from the market portfolio. If, however, $\gamma > \gamma_m$, and no analyst views exist, a positive amount is held in the risk-free asset and $\boldsymbol{w}$ generally differs from $\boldsymbol{w}_m$. Furthermore, (3) shows that the solution is also defined for perfect certainty in views where $\boldsymbol{\Omega} = 0$. Then (5) reduces to $\boldsymbol{w} = \gamma_m/\gamma\boldsymbol{w}_m + 1/\gamma\boldsymbol{P'}(\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P'})^{-1}(\boldsymbol{q} - \boldsymbol{P}\boldsymbol{\pi})]$.

The reverted Black-Litterman model. In contrast to the B-L model, here we assume that an investor's portfolio weight $\boldsymbol{w}$ is known but not the underlying views. For a given confidence in views $\boldsymbol{\Omega}$ – again defined as an $(m \times m)$ diagonal covariance matrix – we are interested in the implicit views $\boldsymbol{P}$ and $\boldsymbol{q}$ that belong to portfolio $\boldsymbol{w}$. An understanding of the exact role of $\boldsymbol{\Omega}$ greatly simplifies this task. Therefore we will look at some technical details of the B-L model in the following.

Note that the probability density function of prior views has the form $\exp\left(-1/2(\boldsymbol{Pr_e} - \boldsymbol{q})'\boldsymbol{\Omega}^{-1}(\boldsymbol{Pr_e} - \boldsymbol{q})\right)$ (compare (2)). Obviously, this function is invariant with respect to the transformation $\boldsymbol{P} \rightarrow \boldsymbol{AP}$, $\boldsymbol{q} \rightarrow \boldsymbol{Aq}$ and $\boldsymbol{\Omega} \rightarrow \boldsymbol{A\Omega A'}$, where $\boldsymbol{A}$ is an $(m \times m)$ matrix.[5] This property allows to scale $\boldsymbol{P}$ such that

$$\boldsymbol{P1_n} = \boldsymbol{1_m}\,, \tag{6}$$

where $\boldsymbol{1}$ is a vector of ones with the appropriate dimension. The matrix of views $\boldsymbol{P}$ can be scaled by choosing $\boldsymbol{A}$ appropriately, making the elements in each row sum up to 1. The reason for the scaling is the following: When the uncertainty $\boldsymbol{\Omega}$ in implicit views is specified, it must obviously be conditioned on the scale of $\boldsymbol{P}$ as for other scales the specified uncertainty would have a different meaning. In the above example the value for $\boldsymbol{\Omega}$ of $(0.003)^2$ may be low for $\boldsymbol{P} = [1\ \ 0\ \ 0\ \ \ldots\ \ 0]$, $q = 0$. However, for the same view given by $\boldsymbol{P} = [0.1\ \ 0\ \ 0\ \ \ldots\ \ 0]$, $q = 0$, the same value of uncertainty may be high. In this example the second specification is only then equivalent to the first if $\boldsymbol{\Omega} = 0.003^2 \cdot 0.1^2$. Consequently, it is necessary to know the scale of $\boldsymbol{P}$ when talking about the associated uncertainty.

The advantage of (6) is that it allows to interpret views (each row i in

[5] In order to keep $\boldsymbol{\Omega}$ diagonal, $\boldsymbol{A}$ must be diagonal. Then $\boldsymbol{A}$ rescales $\boldsymbol{P}$, $\boldsymbol{q}$, and $\boldsymbol{\Omega}$.

$\boldsymbol{P}$) as a portfolio that weights asset j according to the respective view $\boldsymbol{P}_{ij}$ ($j = 1 \dots N$). Moreover, for practical purposes, it facilitates the specification of uncertainty in views when views are thought of as portfolios. Then $\boldsymbol{\Omega}$ can be seen as the variance of a portfolio where the weights are given by implicit views $\boldsymbol{P}_{ij}$. The drawback of this type of fixing the convention is that it is only applicable if views do not sum up to 0 – i.e. if none of the rows in $\boldsymbol{P}$ is orthogonal to $\mathbf{1}$.[6]

The derivation of implicit views is quite straightforward. Let the *active portfolio weights* $\boldsymbol{\delta w}$ be defined as

$$\boldsymbol{\delta w} = \gamma \boldsymbol{w} - \gamma_m \boldsymbol{w}_m \, . \tag{7}$$

Then for the case where only a single view is considered ($m = 1$), (5) can be rearranged to $\boldsymbol{\delta w} = \boldsymbol{P}\alpha$, where α is a scalar. The constraint in (6) is then satisfied when $\alpha = \mathbf{1}'\boldsymbol{\delta w}$. This leads to the *implicit* views given by[7]

$$\boldsymbol{P} = \frac{\boldsymbol{\delta w}'}{\mathbf{1}'\boldsymbol{\delta w}} \, . \tag{8}$$

Hence, $\boldsymbol{P}$ is proportional to active portfolio weights $\boldsymbol{\delta w}$ and depends only on $\boldsymbol{w}$, $\boldsymbol{w}_m$, γ and γ_m. The proportionality constant is determined by the constraint in (6).

Moreover, $\boldsymbol{q}$ takes a value for which $\alpha = \mathbf{1}'\boldsymbol{\delta w}$, hence

$$\boldsymbol{q} = \mathbf{1}'\boldsymbol{\delta w} \left(\frac{1}{\tau}\boldsymbol{\Omega} + \boldsymbol{P\Sigma P}' \right) + \boldsymbol{P\pi} \, . \tag{9}$$

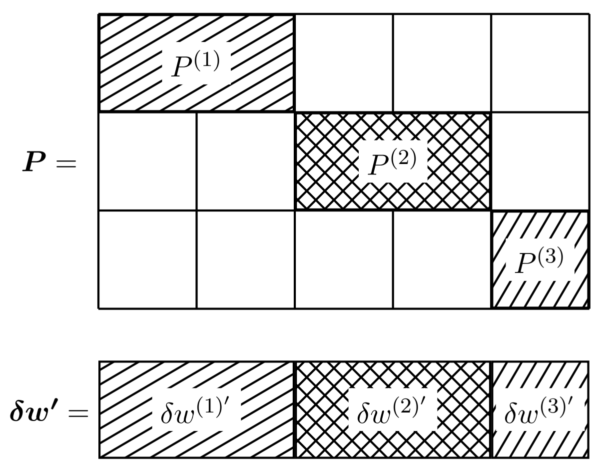
[6] To circumvent this problem one may use the scale $\boldsymbol{PP}' = \mathbb{I}$. While this type of specification is without loss of generality, it makes difficult to obtain an intuition for the uncertainty and implicit views.

[7] In the following we consider the case where $\mathbf{1}'\boldsymbol{\delta w} \neq 0$. This is also possible for $m = 1$ as in this formulation $\boldsymbol{w}$ does not have to sum up to 1. If, however, views $\boldsymbol{P}$ sum up to 0, one may use the constraint $\boldsymbol{PP}' = \mathbb{I}$ instead of (6). Then the scaling equals to the norm of active weights ($\alpha = |\boldsymbol{\delta w}|$).

If there is one view only, (8) and (9) describe the implicit view that belongs to $\boldsymbol{w}$ and the uncertainty specified in $\boldsymbol{\Omega}$.

It is easy to generalize (8) and (9) to more than one implicit view ($m > 1$) as long as views are mutually exclusive, that is, no asset enters in more than one row of $\boldsymbol{P}$. This requires that if $P_{ij} > 0$, then $P_{kj} = 0$ ($i \neq k$; $i, k = 1 \ldots m$; $j = 1 \ldots n$). For these mutually exclusive views, (8) and (9) can be applied with redefined variables. This is illustrated by the following example. Consider a universe of 5 assets, a market portfolio $\boldsymbol{w}_m$ and an investor holding portfolio $\boldsymbol{w}$. Now we assume that there are three uncorrelated and unknown implicit views behind $\boldsymbol{w}$, but the associated uncertainties $\Omega_{1,1}$, $\Omega_{2,2}$, and $\Omega_{3,3}$ are known.

We further assume that the first view is about assets 1 and 2, the second view is about assets 3 and 4 and the third view is solely about asset 5. Then the matrix $\boldsymbol{P}$ and the vector $\boldsymbol{\delta w}$ have the form



where the superscripts (1), (2) and (3) describe the blocks with the respective assets. The white areas in $\boldsymbol{P}$ contain zeros. Due to the assumptions that views are mutually exclusive and uncorrelated ($\boldsymbol{\Omega}$ is diagonal), (8) and (9) can generally be written to

$$\boldsymbol{P}^{(i)} = \frac{\boldsymbol{\delta w^{(i)\prime}}}{\boldsymbol{1'\delta w^{(i)}}} \tag{10}$$

$$\boldsymbol{q} = \left(\frac{1}{\tau}\boldsymbol{\Omega} + \boldsymbol{P\Sigma P'}\right) \begin{pmatrix} \boldsymbol{1'\delta w^{(1)}} \\ \boldsymbol{1'\delta w^{(2)}} \\ \vdots \\ \boldsymbol{1'\delta w^{(m)}} \end{pmatrix} + \boldsymbol{P\pi}\,. \tag{11}$$

for $i = 1 \ldots m$ (in this example $i = 1, 2, 3$). As above, (10) requires that none of the blocks in $\boldsymbol{P}$ sum up to 0.

Thus, when multiple views are allowed in the reverted B-L approach, one additionally has to specify the structure of $\boldsymbol{P}$, i.e. which asset belongs to which block.[8] Then implicit views are given by (10) and (11).

3 Examples

To demonstrate the working of the reverted Black-Litterman model, we consider an example in which the investment universe consists of the 10 MSCI sector indices. To calculate the sample covariance matrix $\boldsymbol{\Sigma}$ we use daily returns calculated in USD from 2000/1 to 2007/8. The covariance matrix is reported in Appendix B. For reverse optimization we set $\boldsymbol{w}_m$ equal to the market capitalizations of the 10 MSCI sectors as of 2007/8 and use an empirically observed value of $\gamma_m = 2.56$. This value is an estimate of the risk premium (excess return divided by variance) over the period 1951-2000 reported in Fama and French (2002).[9] Moreover, we set $\tau = 0.3$. Similar to Drobetz (2002) we find that final results are robust to this choice of τ.

[8] I consider the specification of blocks as natural in the B-L model; analyst teams may form views within their area of knowledge, not sharing responsibility for the consequences of their views with other teams.

[9] See Table I in Fama and French (2002, p.641). Section 4 shows the sensitivity of implicit views when γ_m is changed.

Sector Names	$\boldsymbol{w}$	$\boldsymbol{w_m}$	Diff.	$\boldsymbol{\pi}$ (in %)
ENR	0.35	0.15	0.20	6.33
HC	0.11	0.11	0.00	4.83
IT	0.15	0.15	0.00	9.83
MAT	0.04	0.04	0.00	6.15
FN	0.13	0.13	0.00	6.82
IND	0.12	0.12	0.00	6.59
TEL	0.05	0.05	0.00	6.82
CSTA	0.03	0.13	-0.10	3.59
CDIS	-0.04	0.06	-0.10	6.61
UTI	0.05	0.05	0.00	4.22

Table 1: The vector $\boldsymbol{w}_m$ is the market capitalization weighted portfolio of 10 MSCI sector indices. The observed portfolio $\boldsymbol{w}$ differs from $\boldsymbol{w}_m$ only in the sectors ENERGY, CONSUMER STAPLES and CONSUMER DISCRETIONARY.

3.1 Example 1: Implicit Views

Assume that as a result of the investment process for a client with the same relative risk aversion as the market ($\gamma = \gamma_m = 2.56$) we observe the portfolio weight $\boldsymbol{w}$ given in Table 1.

We consider two examples. In example 1a, we have two analysts; analyst 1 is responsible for forming a view about the sectors ENERGY and MATERIALS while analyst 2 forms a view about all other sectors. The explicit views of analysts 1 and 2 are not known. As in the B-L model, here we also need to specify the uncertainty in analysts' (implicit) views. We assume that both analysts' views $\boldsymbol{q}^{(1)}$ and $\boldsymbol{q}^{(2)}$ lie in the 90% confidence interval of $\pm 20\%$. This implies that $\Omega_{1,1} = \Omega_{2,2} = 1.478 \cdot 10^{-2}$. Note again that we assume that analysts' express their views in the particular way of $\boldsymbol{P}\boldsymbol{1}_n = \boldsymbol{1}_m$ ($n = 10$, $m = 2$).

In example 1b both analysts' implicit views $\boldsymbol{q}$ lie with 90% probability

	Example 1a		Example 1b	
	$\boldsymbol{P}^{(1)}$	$\boldsymbol{P}^{(2)}$	$\boldsymbol{P}^{(1)}$	$\boldsymbol{P}^{(2)}$
ENR	1		1	
HC				
IT				
MAT				
FN				
IND				
TEL				
CSTA		0.5		0.5
CDIS		0.5		0.5
UTI				
$\boldsymbol{q}$ (in %)	10.54	2.50	8.05	4.99

Table 2: This table reports the implicit views in portfolio $\boldsymbol{w}$ given in Table 1. In example 1a there are two analysts whose views $\boldsymbol{q}$ lie with 90% probability within ±20%. Analyst 1 has a view about the ENERGY sector while analyst 2's view is about all remaining MSCI sectors. In example 1b the associated uncertainties are ±2%.

within ±2% ($\Omega_{1,1} = \Omega_{2,2} = 1.478 \cdot 10^{-4}$) Applying (10) and (11) Table 2 reports the implicit views behind portfolio $\boldsymbol{w}$ for both examples.

Example 1a shows that portfolio $\boldsymbol{w}$ with 20% overweight in the ENERGY sector contains the implicit view that ENR is likely[10] to have an expected annual excess return of 10.54% ± 20%. This implicit view is higher than the ENR sector's equilibrium excess return $\boldsymbol{\pi}_1$ (=6.33%) and thus, gives an explanation for the overweight. The second implicit view is that a portfolio with 50% CSTA and 50% CDIS is likely to have an expected excess return of 2.50% ± 20%. This value is lower than the average equilibrium excess return for these two sectors (5.1%). This is seen in (11); higher precision leads to lower expectations $\boldsymbol{q}$ in order to obtain the same portfolio

[10] We will always assume a 90% confidence interval in the following.

$\boldsymbol{w}$. Example 1a shows that in general overweighted (underweighted) assets have implicit views $\boldsymbol{q}$ that are higher (lower) than corresponding equilibrium excess returns in $\boldsymbol{\pi}$.

In example 1b the uncertainty is reduced to $\pm 2\%$. The increase in the views precision leads to values of $\boldsymbol{q}$ (8.05% and 4.99%) that are closer to their equilibrium values (6.33% and 5.1%). In example 1b the overweight in ENR is mostly explained by the high precision (low value of $\Omega_{1,1}$). For this value the implicit view is that the ENR sector's expected annual excess return is 8.05% and the variance is $1.478 \cdot 10^{-4}$. Compared to the ENR sector's historical variance of $5.02 \cdot 10^{-2}$, the precision of the view is very high. This justifies the 20% overweight despite the moderate value of $\boldsymbol{q}_1$. When considering the 10% underweight in CSTA and CDIS one might expect despite the high precision of the view that $\boldsymbol{q}_2$ is clearly below the respective equilibrium excess returns $\boldsymbol{\pi}$. This intuition is wrong; the overweight in ENR, the relatively high correlation (0.56) of ENR with a portfolio of 50% CSTA and CDIS as well as the low variance (0.14) of this portfolio reduce the impact of the underweight on implicit views.[11] Here it is important to see that due to assets' covariances, an analyst's view on one asset may change the allocation of an asset, which another analyst is responsible for. Although both analysts form their views independently, the resulting portfolio weights all include information from both analysts' views.

Examples 1a and 1b show how implicit views differ for different values of uncertainty $\boldsymbol{\Omega}$. The next example shows that implicit views can also be used in the context of portfolio engineering.

[11] Note that while the rows of $\boldsymbol{P}$ are independent, $\boldsymbol{q}_i$ in general depends on the jth row of $\boldsymbol{P}$ $(i \neq j)$ and on assets' covariances. This comes from the term $\boldsymbol{P\Sigma P'}$ in (11).

	Example 2		
Sector Names	$\boldsymbol{w}^{RA}$	$\boldsymbol{w}$	Diff.
ENR	0.22	0.35	-0.13
HC	0.07	0.11	-0.04
IT	0.09	0.15	-0.05
MAT	0.03	0.04	-0.02
FN	0.08	0.13	-0.05
IND	0.08	0.12	-0.04
TEL	0.03	0.05	-0.02
CSTA	0.02	0.03	-0.01
CDIS	-0.02	-0.04	0.01
UTI	0.03	0.05	-0.02

Table 3: Portfolio $\boldsymbol{w}^{RA}$ incorporates the implicit Black-Litterman views of portfolio $\boldsymbol{w}$, but it has a different risk aversion ($\gamma = 4$ instead of $\gamma = 2.56$.)

3.2 Example 2: Change in Views vs. Change in Risk Aversion

Portfolio $\boldsymbol{w}$ in Table 1 is specified for an investor with relative risk aversion $\gamma = 2.56$. Assume that within the same product family[12] one sells a less risky portfolio $\boldsymbol{w}^{RA}$ for more risk averse clients having $\gamma = 4$. At the same time one wishes to adopt the same investment views as in $\boldsymbol{w}$. Assume further, that the views behind $\boldsymbol{w}$ are not known. Thus, in order to use the Black-Litterman model, implicit views first have to be extracted from $\boldsymbol{w}$. This can be done applying (10) and (11) when the uncertainty $\boldsymbol{\Omega}$ and the structure of $\boldsymbol{P}$ are known. In the following we use the input specifications of example 1a.

The implicit views for the specifications in example 1a reported in Table 2 can be plugged into the Black-Litterman model (5) with $\gamma = 4$ to obtain the less risky portfolio $\boldsymbol{w}^{\boldsymbol{RA}}$. The resulting portfolio is shown in

[12] Products with some common properties such as similar investment restrictions, the same type of clients, the same asset universe etc.

Table 3. Note that $\boldsymbol{w}^{RA}$ does not sum up to 1 and the remaining wealth is invested into the riskfree asset. Moreover, the ENR sector's weight is strongly decreased compared to $\boldsymbol{w}$ while the weights of CSTA and CDIS are virtually unchanged. The reason for this can be seen in (5): an increase of γ decreases the investment into $\boldsymbol{w}_m$ (the first summand) and also decreases the view based investment (second summand). These two effects both decrease the allocation in the ENR sector compared to $\boldsymbol{w}$ and offset each other for the sectors CSTA and CDIS. Note also that in contrast to $\boldsymbol{w}$ ($\gamma = 2.56$) where only the allocations of the sectors ENR, CSTA and CDIS differ from $\boldsymbol{w}_m$, for $\boldsymbol{w}^{\boldsymbol{RA}}$ ($\gamma = 4$), all allocations differ. This is also due to the first summand in (5) which reduces the allocations of all weights when γ is increased.

Example 2 shows that even when views behind portfolio $\boldsymbol{w}$ are not known in advance, for a given specification of $\boldsymbol{\Omega}$ and $\boldsymbol{P}$, implicit views can be extracted and used to construct a portfolio with a different (here: lower) risk. For products within the same product family this avoids the problem that products with different risk profiles are based on different views.

4 Sensitivity Test

The previous examples have used an empirical estimate γ_m of 2.56 taken from Fama and French (2002). However, this value differs for different time periods, i.e. Fama and French (2002) estimate $\gamma_m = 1.25$ for the period 1872-1950 and $\gamma_m = 1.71$ for 1872-2000.[13]

As implicit views $\boldsymbol{P}$ and $\boldsymbol{q}$ depend on γ_m and γ, it is important to know their sensitivity with respect to changes in these two variables. The upper

[13]The Fama-French estimates are based on the S&P 500.

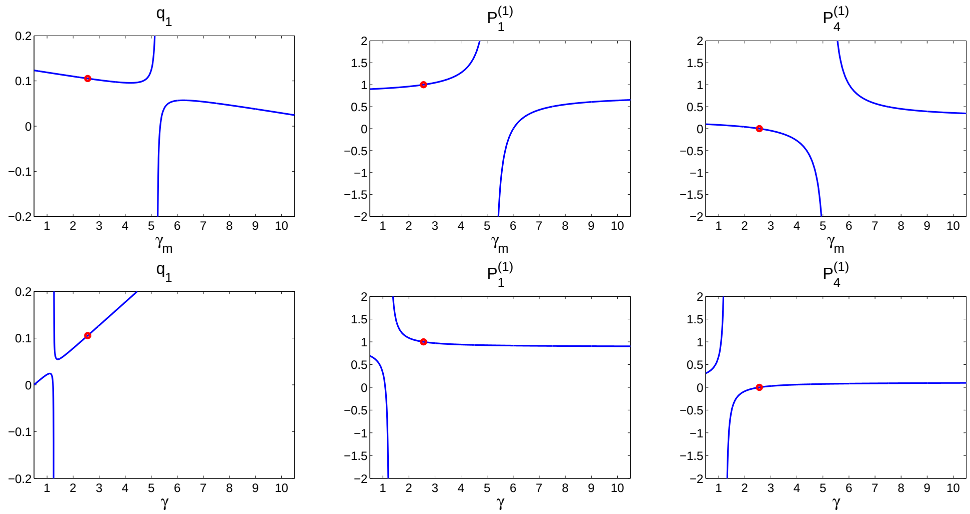


Figure 1: The upper plots show the relation between the implicit views of example 1a and the market risk aversion γ_m. The lower plots show the same relation for a client's risk aversion γ.

plots in Figure 1 show how results of example 1a change when γ_m is varied while keeping γ constant ($= 2.56$). The solutions of example 1a ($\boldsymbol{q}_1 = 10.54$, $\boldsymbol{P}_1^{(1)} = 1$, $\boldsymbol{P}_4^{(1)} = 0$) are marked with a circle. Figure 1 shows that except for the vicinity of $\gamma = 5.21$ implicit views are robust. For $\gamma = 5.21$ the sum of $\boldsymbol{\delta w^{(1)}}$ becomes zero which is the problem discussed in Section 2. The less intuitive specification of $\boldsymbol{PP'} = \mathbb{I}$ in (6) would avoid such extreme points.

For varying values of γ and constant γ_m ($= 2.56$), the lower plots in Figure 1 show resulting implicit views. The extremum is at $\gamma = 1.26$. Around this value implicit views are extremely sensitive to changes in γ. Moreover, Figure 1 shows that in example 1a when γ_m is above 7 and γ above 3, the values in $\boldsymbol{P}$ are robust and only $\boldsymbol{q}$ changes.[14]

More generally, views $\boldsymbol{P}^{(i)}$ and $\boldsymbol{q}_i$ are then very sensitive to changes

[14]In a more general version of the Black-Litterman model Giacometti, Bertocchi, Rachev, and Fabozzi (2007) consider values for γ_m up to 36. Here, however, qualitative results would not differ for such high values.

whenever γ/γ_m is close to

$$\frac{\sum_{j\in\mathbb{S}^{(i)}} \boldsymbol{w}_{m,j}}{\sum_{j\in\mathbb{S}^{(i)}} \boldsymbol{w}_j},$$

where $\mathbb{S}^{(i)}$ denotes the set of assets $j = 1\ldots n$ where $\boldsymbol{P}_{ij} > 0$. If γ/γ_m becomes close to the above ratio, one should be careful with the interpretation of $\boldsymbol{P}$ and $\boldsymbol{q}$ or use the specification $\boldsymbol{PP'} = \mathbb{I}$ instead of (6). Then the curves in Figure 1 are smooth.

It is also of interest to know how $\boldsymbol{q}$ changes when the uncertainty $\boldsymbol{\Omega}$ is changed. From (11) it is clear that higher $\boldsymbol{\Omega}$ increases $\boldsymbol{q}$ as larger uncertainty has to come together with a larger view in order to obtain the same portfolio $\boldsymbol{w}$. The impact of an increase of uncertainty on $\boldsymbol{q}$ is given by differentiation:

$$\boldsymbol{\Delta q_i} = \boldsymbol{\Delta\Omega_{i,i}}\frac{\boldsymbol{1'\delta w}^{(i)}}{\tau}.$$

In example 1a and 1b the derivatives are equal to 1.707 for $\boldsymbol{q}_1$ and -1.707 for $\boldsymbol{q}_2$. In order to obtain portfolio $\boldsymbol{w}$ an increase of the views variance by 1% must thus come together with an increase of $\boldsymbol{q}_1$ by 1.707% and a decrease of $\boldsymbol{q}_2$ by 1.707%.

5 Conclusion

In practice, portfolios are often formed using different quantitative and qualitative models on different hierarchical levels (such as strategic and tactical asset allocation and stock selection levels). When not only the Black-Litterman model is applied to construct a portfolio, views about future market movements are not explicitly known. This makes it difficult to consistently analyze and to communicate final views that are implicit in a portfolio. Moreover, implicit views must be known in order to be able to change a portfolio's risk but not the views.

This paper shows that by a rearrangement of the Black-Litterman model a set of implicit views can be extracted from portfolio weights. A convenient property of these implicit views is that each of them can be interpreted as the expected return and standard deviation of a portfolio.

In order to extract implicit views from portfolios one has to specify the uncertainty and structure of views. The structure determines the number of views in a portfolio and the assets that belong to each view. Here it is assumed that views about assets are mutually exclusive, that is, no asset can enter into more than one view.

Finally, this paper provides two examples that are relevant to practitioners. The first example shows the working of the reverted Black-Litterman model; thus, how to extract implicit views from a portfolio. The second example shows how implicit views from a portfolio can be implemented for a portfolio with a different level of risk.

A Derivation of the Black-Litterman Weights

The Black-Litterman weights in (5) can be derived in the following way. Plugging (3) into (4) yields

$$\boldsymbol{w} = \frac{1}{\gamma}\boldsymbol{\Sigma^{-1}}\left(\boldsymbol{\Sigma^{-1}} + \tau \boldsymbol{P'\Omega^{-1}P}\right)^{-1}\left(\boldsymbol{\Sigma^{-1}\pi} + \tau\boldsymbol{P'\Omega^{-1}q}\right)$$

Using the identity $(\boldsymbol{A}+\boldsymbol{XX'})^{-1} = \boldsymbol{A^{-1}} - \boldsymbol{A^{-1}X}(\mathbb{I}+\boldsymbol{X'A^{-1}X})^{-1}\boldsymbol{X'A^{-1}}$ one can write

$$\gamma\boldsymbol{w} = \left[1 - \tau\boldsymbol{P'\Omega^{-1/2}}(\mathbb{I}+\boldsymbol{u})^{-1}\boldsymbol{\Omega^{-1/2}P\Sigma}\right]\left[\boldsymbol{\Sigma^{-1}\pi} + \tau\boldsymbol{P'\Omega^{-1}q}\right]$$

with $\boldsymbol{u} = \tau\boldsymbol{\Omega^{-1/2}P\Sigma P'\Omega^{-1/2}}$. Together with the equilibrium excess returns $\boldsymbol{\pi} = \gamma_m\boldsymbol{\Sigma w}_m$ we obtain

$$\begin{aligned}
\gamma\boldsymbol{w} = \gamma_m\boldsymbol{w}_m &\\
&- \tau\boldsymbol{P'\Omega^{-1/2}}(\mathbb{I}+\boldsymbol{u})^{-1}\boldsymbol{\Omega^{-1/2}P\pi}\\
&+ \tau\boldsymbol{P'\Omega^{-1}q}\\
&- \tau^2\boldsymbol{P'\Omega^{-1/2}}(\mathbb{I}+\boldsymbol{u})^{-1}\boldsymbol{\Omega^{-1/2}P\Sigma P'\Omega^{-1}q}\\
= \gamma_m\boldsymbol{w}_m &\\
&- \tau\boldsymbol{P'\Omega^{-1/2}}(\mathbb{I}+\boldsymbol{u})^{-1}\boldsymbol{\Omega^{-1/2}P\pi}\\
&+ \tau\boldsymbol{P'\Omega^{-1/2}}\mathbb{I}\boldsymbol{\Omega^{-1/2}q}\\
&- \tau\boldsymbol{P'\Omega^{-1/2}}(\mathbb{I}+\boldsymbol{u})^{-1}\boldsymbol{u\Omega^{-1/2}q}\\
= \gamma_m\boldsymbol{w}_m &\\
&- \tau\boldsymbol{P'\Omega^{-1/2}}(\mathbb{I}+\boldsymbol{u})^{-1}\boldsymbol{\Omega^{-1/2}P\pi}\\
&+ \tau\boldsymbol{P'\Omega^{-1/2}}(\mathbb{I}+\boldsymbol{u})^{-1}\boldsymbol{\Omega^{-1/2}q}\\
= \gamma_m\boldsymbol{w}_m &+ \tau\boldsymbol{P'\Omega^{-1/2}}(\mathbb{I}+\boldsymbol{u})^{-1}\boldsymbol{\Omega^{-1/2}}(\boldsymbol{q}-\boldsymbol{P\pi})\,.
\end{aligned}$$

Together with $(\mathbb{I}+\boldsymbol{u})^{-1} = (\mathbb{I}+\tau\boldsymbol{\Omega^{-1/2}P\Sigma P'\Omega^{-1/2}})^{-1} = \boldsymbol{\Omega^{1/2}}(\boldsymbol{\Omega}+\tau\boldsymbol{P\Sigma P'})^{-1}\boldsymbol{\Omega^{1/2}}$ we obtain the key equation of the Black-Litterman model

$$\gamma\boldsymbol{w} = \gamma_m\boldsymbol{w} + \tau\boldsymbol{P'}(\boldsymbol{\Omega}+\tau\boldsymbol{P\Sigma P'})^{-1}(\boldsymbol{q}-\boldsymbol{P\pi})\,.$$

B Covariance Matrix

Examples 1 and 2 use daily returns (in USD) from 10 MSCI sector indices for the period 2000/1 to 2007/8. The covariance matrix (times 100) and sector names are reported below

$$\begin{array}{l} \text{ENR} \\ \text{HC} \\ \text{IT} \\ \text{MAT} \\ \text{FN} \\ \text{IND} \\ \text{TEL} \\ \text{CSTA} \\ \text{CDIS} \\ \text{UTI} \end{array} \quad \begin{bmatrix} 5.02 & & & & & & & & & \\ 1.84 & 2.63 & & & & & & & & \\ 2.22 & 2.10 & 9.49 & & & & & & & \\ 2.54 & 1.58 & 3.06 & 3.51 & & & & & & \\ 2.32 & 1.98 & 3.81 & 2.69 & 3.40 & & & & & \\ 2.08 & 1.82 & 4.05 & 2.73 & 2.85 & 3.20 & & & & \\ 2.21 & 1.85 & 4.22 & 2.46 & 2.98 & 2.60 & 4.76 & & & \\ 1.45 & 1.40 & 1.33 & 1.37 & 1.53 & 1.37 & 1.39 & 1.47 & & \\ 1.99 & 1.71 & 4.29 & 2.66 & 2.90 & 2.93 & 2.81 & 1.33 & 3.36 & \\ 1.97 & 1.38 & 1.70 & 1.67 & 1.79 & 1.58 & 1.69 & 1.18 & 1.53 & 2.22 \end{bmatrix}$$

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